Back to the chain matrix:

$$V_1 = AV_2 - BI_2 \qquad I_1 = CV_2 - DI_2 \tag{5-24}$$

A, B, C, and D are called the *chain parameters;* their matrix

$$\mathbf{T} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{5-25}$$

$$A = \left(\frac{V_1}{V_2}\right)_{I_2=0} \qquad B = \left(\frac{V_1}{-I_2}\right)_{V_2=0} \qquad C = \left(\frac{I_1}{V_2}\right)_{I_2=0} \qquad D = \left(\frac{I_1}{-I_2}\right)_{V_2=0} \tag{5-27}$$

Hence, **T** can be found from the schemes shown in Fig. 5-11.

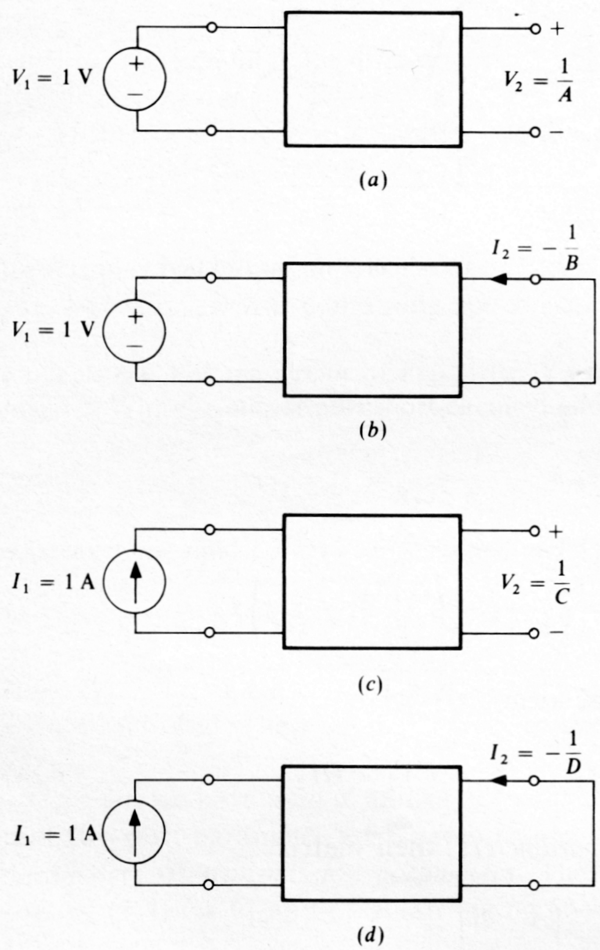
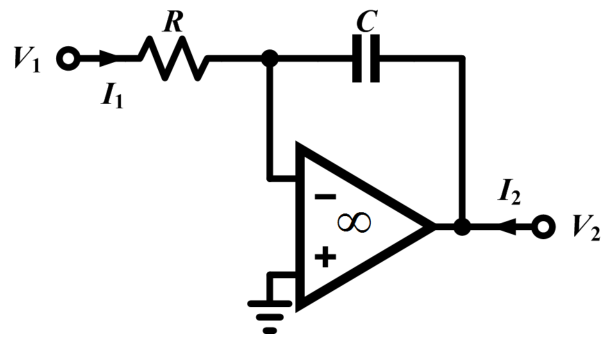


Figure 5-11 Schemes for calculating the chain parameters of a two-port.

Integrator:



$$V_2 = \frac{-V_1}{sRC} \qquad I_1 = \frac{V_1}{R}$$

$V_1 = -sRCV_2 + 0.I_2$
$I_1 = V_1 / R = -sCV_2 + 0.I_2$

$$T = \begin{bmatrix} -sRC & 0 \\ -sC & 0 \end{bmatrix} = sC\begin{bmatrix} R & 0 \\ 1 & 0 \end{bmatrix}$$

Cascaded integrators:

$$T_1T_2 = s^2C_0^2\begin{bmatrix} R^2 & 0 \\ R & 0 \end{bmatrix} = (-sRC_0)^2\begin{bmatrix} 1 & 0 \\ 1/R & 0 \end{bmatrix}$$

NOTE: B=D=0 means buffered two-port!

Example 5-3 Find **Y** and **Z** for the circuit of Fig. 5-7*a*.

From Fig. 5-7*b* (where we have chosen for simplicity $V_1 = 1$ V), by inspection

$$y_{11} \triangleq \frac{I_1^1}{V_1} = I_1^1 = sC_1 + \frac{1}{R} \qquad y_{21} \triangleq \frac{I_2^1}{V_1} = I_2^1 = -\frac{1}{R}$$

and from Fig. 5-7*c*, with $V_2 = 1$ V,

$$y_{12} \triangleq \frac{I_1^2}{V_2} = I_1^2 = -\frac{1}{R} \qquad y_{22} \triangleq \frac{I_2^2}{V_2} = I_2^2 = sC_2 + \frac{1}{R}$$

Hence

$$\mathbf{Y} = \begin{bmatrix} sC_1 + \frac{1}{R} & -\frac{1}{R} \\ -\frac{1}{R} & sC_2 + \frac{1}{R} \end{bmatrix}$$

Therefore

$$\Delta_Y = \left(sC_1 + \frac{1}{R}\right)\left(sC_2 + \frac{1}{R}\right) - \frac{1}{R^2} = s^2 C_1 C_2 + \frac{s(C_1 + C_2)}{R}$$

and, by (5-17),

$$\mathbf{Z} = \begin{bmatrix} \frac{sC_2 + 1/R}{\Delta_Y} & \frac{1/R}{\Delta_Y} \\ \frac{1/R}{\Delta_Y} & \frac{sC_1 + 1/R}{\Delta_Y} \end{bmatrix}$$

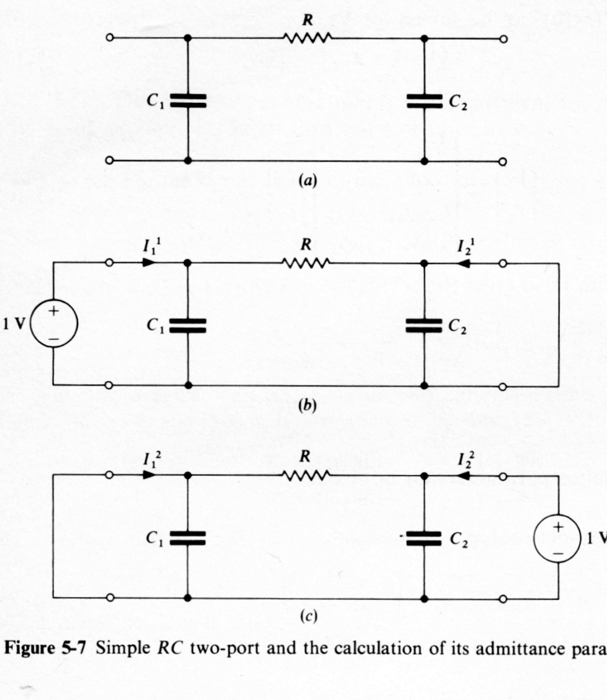


Figure 5-7 Simple *RC* two-port and the calculation of its admittance parameters.

Of course, **Z** can also be obtained directly, using the method illustrated in Fig. 5-4. For our example, this gives

$$z_{11} = \frac{V_1^1}{I_1} = \frac{1}{sC_1 + \dfrac{1}{R + 1/sC_2}}$$

$$z_{11} = \frac{sRC_2 + 1}{s^2 RC_1 C_2 + s(C_1 + C_2)} = \frac{sC_2 + 1/R}{s^2 C_1 C_2 + s(C_1 + C_2)/R}$$

Example 5-4 Find **Z** and **Y** for the two-port shown in Fig. 5-8.

With an input current $I_1 = 1$ A, from (5-2) and Fig. 5-4*b*,

$$z_{11} = V_1^1 = \frac{sL_1 R_1}{sL_1 + R_1} + \frac{R_2/sC}{R_2 + 1/sC} = \frac{s}{s+1} + \frac{1/s}{1 + 1/s} = 1$$

and

$$z_{21} = V_2^1 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1/s}{1 + 1/s} = \frac{1}{s+1}$$

Similarly, from (5-3) and Fig. 5-4*c*,

$$z_{12} = V_1^2 = \frac{R_2/sC}{R_2 + 1/sC} = \frac{1}{s+1}$$

and

$$z_{22} = V_2^2 = \frac{sL_2 R_3}{sL_2 + R_3} + \frac{R_2/sC}{R_2 + 1/sC} = 1$$

Hence,

$$\Delta_Z = 1 - \frac{1}{(s+1)^2} = \frac{s^2 + 2s}{s^2 + 2s + 1}$$

and by (5-19)

$$y_{11} = \frac{z_{22}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

$$y_{12} = y_{21} = -\frac{z_{12}}{\Delta_Z} = -\frac{s+1}{s^2 + 2s}$$

$$y_{22} = \frac{z_{11}}{\Delta_Z} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

Again, y_{11} will be checked using the scheme of Fig. 5-5*b*. For $V_1 = 1$ V,

$$y_{11} = I_1^1 = \frac{1}{\dfrac{sL_1 R_1}{sL_1 + R_1} + \dfrac{1}{1/R_2 + sC + 1/R_3 + 1/sL_2}}$$

$$= \frac{1}{\dfrac{s}{s+1} + \dfrac{1}{2 + s + 1/s}} = \frac{s^2 + 2s + 1}{s^2 + 2s}$$

as before.

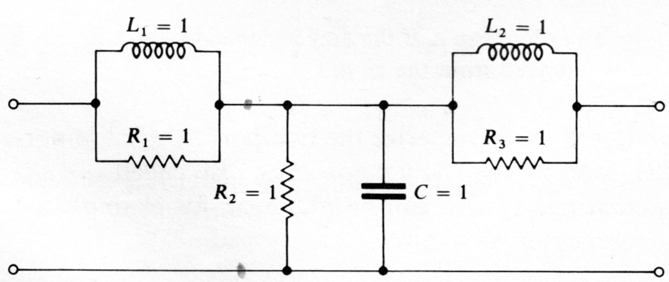


Figure 5-8 *RLC* two-port example.

Symmetric, reciprocal constant-resistance twoport.

Example 5-5 Find the z_{ij} for the two-port shown in Fig. 5-9.
From Fig. 5-4, we obtain

$$z_{11} = Z_1 + \frac{Z_2(Z_3 + Z_4)}{Z_2 + Z_3 + Z_4}$$

$$z_{12} = z_{21} = \frac{Z_2 Z_4}{Z_2 + Z_3 + Z_4}$$

$$z_{22} = \frac{Z_4(Z_2 + Z_3)}{Z_2 + Z_3 + Z_4}$$

The reader should fill in the details.

Hitherto, all two-port examples contained reciprocal circuits. Hence, by (5-5), $z_{12} = z_{21}$ held, and so did

$$y_{12} = y_{21} \tag{5-21}$$

Equation (5-21) can be obtained either from Eqs. (5-5) and (5-19) or from Fig. 1-6*b*. For circuits containing active elements, (5-5) and (5-21) need not hold.

Example 5-6 For the circuit of Fig. 5-10*a*, **Y** can readily be found using the scheme of Fig. 5-5. With the output port short-circuited and $V_1 = 1$ V (Fig. 5-10*b*),

$$y_{11} = I_1^1 = \frac{1}{R_1} + \frac{1}{R_2} \qquad y_{21} = I_2^1 = -\frac{G}{R_3} - \frac{1}{R_2}$$

whereas if the input port is short-circuited and $V_2 = 1$ V (Fig. 5-10*c*),

$$y_{12} = I_1^2 = -\frac{1}{R_2} \qquad \text{and} \qquad y_{22} = \frac{1}{R_3} + \frac{1}{R_2}$$

Here, $y_{12} \neq y_{21}$, except if $G = 0$ or $R_3 \to \infty$, that is, if the active element (which here is a voltage-controlled voltage source) is removed from the circuit.

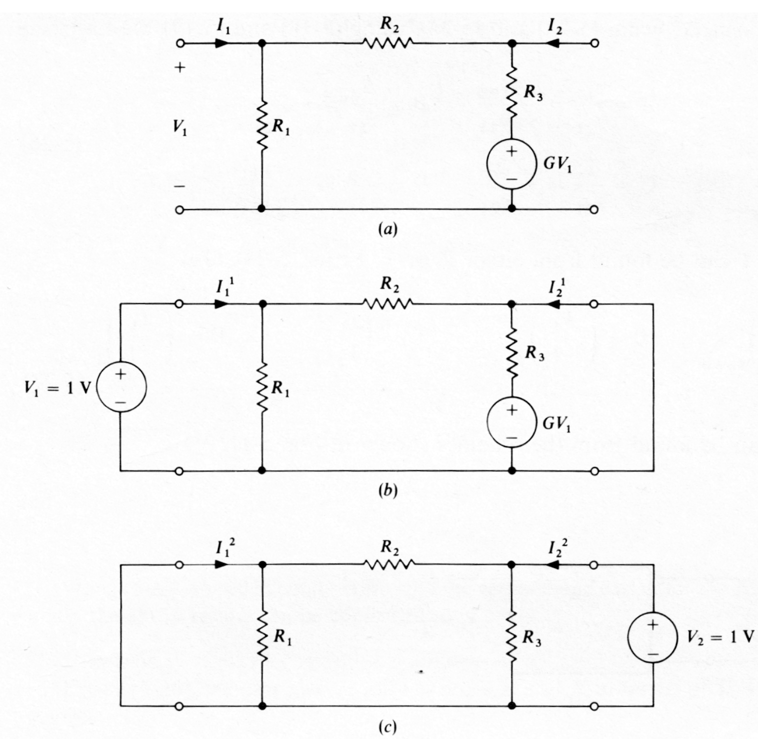


Figure 5-10 (*a*) Nonreciprocal two-port; (*b*) and (*c*) calculation of **Y** for the two-port.

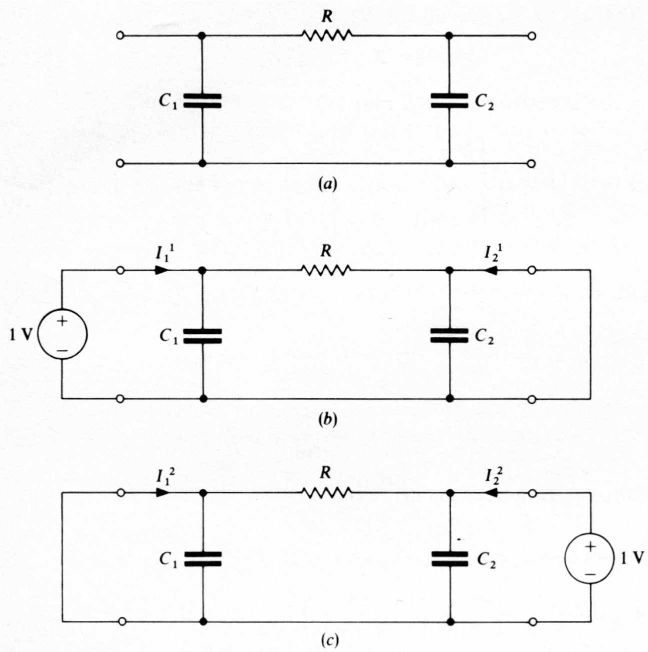


Figure 5-7 Simple *RC* two-port and the calculation of its admittance parameters.

Example 5-7 For the two-port of Fig. 5-7*a*, using Fig. 5-11, we find

$$A = \left(\frac{1}{V_2}\right)_{\substack{V_1 = 1\text{ V}\\ I_2 = 0}} = \left(\frac{1/sC_2}{R + 1/sC_2}\right)^{-1} = sRC_2 + 1$$

$$B = \left(-\frac{1}{I_2}\right)_{\substack{V_1 = 1\text{ V}\\ V_2 = 0}} = R$$

$$C = \left(\frac{1}{V_2}\right)_{\substack{I_1 = 1\text{ A}\\ I_2 = 0}} = \left(\frac{1}{V_1 \dfrac{1/sC_2}{R + 1/sC_2}}\right)_{\substack{I_1 = 1\text{ A}\\ I_2 = 0}}$$

$$= \frac{1}{\dfrac{(1/sC_1)(R + 1/sC_2)}{(1/sC_1 + R + 1/sC_2)} \dfrac{1/sC_2}{R + 1/sC_2}} = s^2RC_1C_2 + s(C_1 + C_2)$$

and

$$D = \left(-\frac{1}{I_2}\right)_{\substack{I_1 = 1\text{ A}\\ V_2 = 0}} = \left(\frac{1/R}{sC_1 + 1/R}\right)^{-1} = sRC_1 + 1$$

When (5-26) is used in conjunction with the values found earlier for the **Z** and **Y** of this circuit, the above results can be confirmed.

Check:

Example 5-8 For the circuit of Fig. 5-7*a*,

$$\Delta_T = (sRC_2 + 1)(sRC_1 + 1) - R(s^2RC_1C_2 + sC_1 + sC_2) = 1$$

as expected.

Example 5-9 For the circuit of Fig. 5-10*a*, the chain parameters can be found from the admittance parameters calculated earlier, for example:

$$A = -\frac{y_{22}}{y_{21}} = -\frac{1/R_3 + 1/R_2}{-G/R_3 - 1/R_2} = \frac{R_2 + R_3}{GR_2 + R_3}$$

$$B = -\frac{1}{y_{21}} = -\frac{1}{-G/R_3 - 1/R_2} = \frac{R_2 R_3}{GR_2 + R_3}$$

$$C = -\frac{y_{11}y_{22} - y_{12}y_{21}}{y_{21}} = -\frac{y_{11}y_{22}}{y_{21}} + y_{12}$$

$$= -\frac{(1/R_1 + 1/R_2)(1/R_3 + 1/R_2)}{-G/R_3 - 1/R_2} - \frac{1}{R_2}$$

$$= \frac{(R_1 + R_2)(R_2 + R_3)}{(GR_2 + R_3)R_1 R_2} - \frac{1}{R_2}$$

$$D = -\frac{y_{11}}{y_{21}} = -\frac{1/R_1 + 1/R_2}{-G/R_3 - 1/R_2} = \frac{(R_1 + R_2)R_3}{R_1(GR_2 + R_3)}$$

Hence

$$AD - BC = \frac{\dfrac{(R_2 + R_3)(R_1 + R_2)R_3}{R_1} - R_2 R_3\left[\dfrac{(R_1 + R_2)(R_2 + R_3)}{R_1 R_2} - \dfrac{GR_2 + R_3}{R_2}\right]}{(GR_2 + R_3)^2}$$

or, after simplifications,

$$AD - BC = \frac{R_3}{GR_2 + R_3}$$

Hence, $AD - BC = 1$ holds only if $R_3 \to \infty$ or $G \to 0$, that is, only if the controlled source is removed from the circuit.

Since Eq. (5-4) can be rearranged six different ways with two parameters on the left-hand side and two on the right-hand side, we can define six different sets of two-port parameters. The reader should consult Ref. 1, table 17-1, for a listing of these parameters and for the formulas needed to convert from one set to another.

Transfer Functions for Terminated Twoports

Transfer Function: Possible output/known input (voltage or current. For a two-port, we may have

Voltage ratio, voltage gain: $A_v(s) = V_s(s) / E(s)$

Transfer Admittance: $Y_T(s) = I_2(S) / E(s)$

Transfer Impedance: $Z_T(s) = V_2(s) / I(s)$

Current Ratio, current gain: $A_I(s) = I_2(s) / I(s)$

5-5 TRANSFER FUNCTIONS

When the two-port is excited by a generator and terminated by a load, as illustrated in Fig. 5-1, the signal-transfer properties of the *complete* circuit can be described by an appropriately chosen *transfer function.* We define the transfer function as the ratio of an output variable (voltage or current) to a known input quantity (generator voltage or current).† Since most practical generator and load impedances are essentially resistive, we shall restrict our discussions to resistor-terminated reactance two-ports.

In the simplest (and least useful) situation, both terminations are zero or infinite. Then we have one of the four configurations depicted in Fig. 5-18. These circuits are called *unterminated* (*unloaded*) two-ports. The proper choice of a transfer function for any of these circuits is obvious and unique. For example, for the circuit of Fig. 5-18*a*, the output variable must be V_2 since $I_2 \equiv 0$; the known input quantity is the generator voltage E. Hence, we must choose the *voltage ratio* A_V, defined by

$$A_V(s) \triangleq \frac{V_2(s)}{E(s)} \tag{5-84}$$

[The reader should keep in mind the dual interpretation of the variable s. Thus, for $s = j\omega$, $A_V(j\omega)$ may represent the ratio of the steady-state sine-wave voltage phasors at the output and input. In general, however, $A_V(s)$ is the ratio of the Laplace-transformed output signal $v_2(t)$ and generator signal $e(t)$ for a two-port initially free of stored energy.]

Similarly, for the circuit in Fig. 5-18*b* the transfer function must be the *transfer admittance*

$$Y_T(s) \triangleq \frac{I_2(s)}{E(s)} \tag{5-85}$$

For the circuit of Fig. 5-18*c*, the transfer function is the *transfer impedance*

$$Z_T(s) \triangleq \frac{V_2(s)}{I(s)} \tag{5-86}$$

† Here all quantities are assumed to be functions of s, not t.

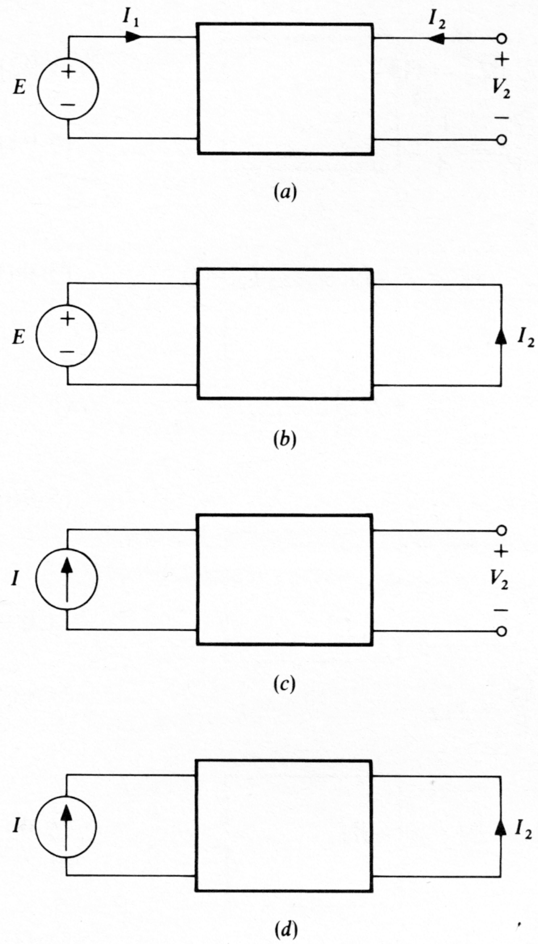


Figure 5-18 Unterminated two-ports.

Finally, for the circuit of Fig. 5-18*d*, the transfer function is the *current ratio*

$$A_I(s) \triangleq \frac{I_2(s)}{I(s)} \tag{5-87}$$

The transfer functions can readily be calculated from the two-port parameters **Z** or **Y** or **T**. For example, for the circuit of Fig. 5-18*a*, from (5-9)

$$V_1 = z_{11}I_1 + z_{12}I_2 = z_{11}I_1 \qquad V_2 = z_{12}I_1 + z_{22}I_2 = z_{12}I_1 \tag{5-88}$$

Hence

$$A_V = \frac{V_2}{E} = \frac{V_2}{V_1} = \frac{z_{12}}{z_{11}} \tag{5-89}$$

Alternatively, from (5-10)

$$I_2 = y_{12}V_1 + y_{22}V_2 = 0 \tag{5-90}$$

which gives

$$A_V = \frac{V_2}{V_1} = -\frac{y_{12}}{y_{22}} \tag{5-91}$$

Or, from (5-24),

$$V_1 = AV_2 - BI_2 = AV_2 \tag{5-92}$$

so that

$$A_V = \frac{V_2}{V_1} = \frac{1}{A} \tag{5-93}$$

For the circuit of Fig. 5-18*b*, from (5-9),

$$V_1 = z_{11}I_1 + z_{12}I_2 = E \qquad V_2 = z_{12}I_1 + z_{22}I_2 = 0 \tag{5-94}$$

Solving (5-94) for I_2 gives

$$I_2 = -\frac{Ez_{12}}{z_{11}z_{22} - z_{12}^2} = -E\frac{z_{12}}{\Delta_Z} \tag{5-95}$$

so that

$$Y_T \triangleq \frac{I_2}{E} = -\frac{z_{12}}{\Delta_Z} \tag{5-96}$$

Alternatively, from (5-10)

$$I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E \tag{5-97}$$

so that

$$Y_T = \frac{I_2}{E} = y_{12} \tag{5-98}$$

Or, from (5-24),

$$V_1 = E = AV_2 - BI_2 = -BI_2 \tag{5-99}$$

which gives

$$Y_T = \frac{I_2}{E} = -\frac{1}{B} \tag{5-100}$$

directly.

Exactly analogous manipulations give

$$Z_T = \frac{V_2}{I} \triangleq z_{12} = -\frac{y_{12}}{\Delta_Y} = \frac{1}{C} \tag{5-101}$$

for the circuit of Fig. 5-18*c* and

$$A_I = -\frac{z_{12}}{z_{22}} = \frac{y_{12}}{y_{11}} = -\frac{1}{D} \tag{5-102}$$

for the circuit of Fig. 5-18*d*.

If the two-port has a single resistive termination, it is called a *singly terminated* or *singly loaded* two-port. Four possible circuits for such a two-port are illustrated in Fig. 5-19. Notice that two other possible circuits exist which may be obtained by replacing the generator and its internal impedance R by its Norton equivalent in the circuit of Fig. 5-19*b* or by its Thevenin equivalent in Fig. 5-19*d*. Their transfer functions differ only by a factor R from those of Fig. 5-19*b* and *d*, and hence they do not merit separate treatment.

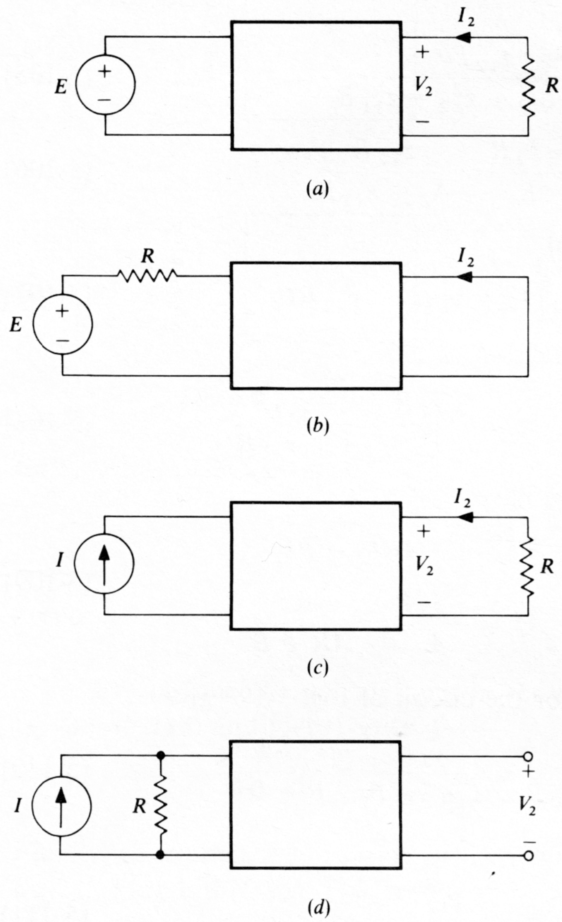


Figure 5-19 Singly terminated two-ports.

For the circuit of Fig. 5-19*a*, we can choose $A_V = V_2/E$ as the transfer function. (Again, a trivially different choice is to select $Y_T \triangleq I_2/E$; since $I_2 = -V_2/R$, here $Y_T = -A_V/R$.)

For the circuit of Fig. 5-19*b*, the transfer function may be selected as $Y_T = I_2/E$ [or as $A_I = I_2/I = I_2/(E/R) = RY_T$ if the Norton generator model is substituted].

For the circuit of Fig. 5-19*c*, we can use $A_I = I_2/I$ or, as trivial variant, $Z_T = V_2/I = -I_2R/I = -RA_I$. For the circuit of Fig. 5-19*d*, the transfer function can be $Z_T = V_2/I$ or if the Thevenin equivalent is used for the generator, $A_V = V_2/E = V_2/(IR) = Z_T/R$.

The transfer functions of the singly loaded two-port can also easily be found in terms of the two-port parameters and R, as will be shown next. For the circuit of Fig. 5-19*a*, combining the branch relations

$$V_1 = E \qquad V_2 = -RI_2 \tag{5-103}$$

and the two-port relations (5-9), we get

$$z_{11}I_1 + z_{12}I_2 = E \qquad z_{12}I_1 + (z_{22} + R)I_2 = 0 \tag{5-104}$$

which gives

$$I_2 = \frac{z_{12}E}{-z_{11}z_{22} + z_{12}^2 - z_{11}R} \tag{5-105}$$

so that

$$A_V = \frac{V_2}{E} = \frac{-I_2R}{E} = \frac{z_{12}R}{\Delta_Z + z_{11}R} \tag{5-106}$$

Alternatively, from (5-103) and (5-10)

$$I_2 = y_{12}V_1 + y_{22}V_2 = y_{12}E - y_{22}RI_2 \tag{5-107}$$

which gives

$$I_2 = \frac{y_{12}E}{1 + y_{22}R} \qquad A_V = \frac{-I_2R}{E} = \frac{-y_{12}R}{1 + y_{22}R} \tag{5-108}$$

Finally, from (5-24), using (5-103), we have

$$V_1 = E = AV_2 - BI_2 = -ARI_2 - BI_2$$
$$I_2 = \frac{-E}{AR + B} \qquad A_V = \frac{-I_2R}{E} = \frac{R}{AR + B} \tag{5-109}$$

Similar calculations performed for the circuit of Fig. 5-19*b* give

$$Y_T \triangleq \frac{I_2}{E} = \frac{-z_{12}}{\Delta_Z + z_{22}R} = \frac{y_{12}}{1 + y_{11}R} = \frac{-1}{B + DR} \tag{5-110}$$

For the circuit of Fig. 5-19*c*,

$$A_I \triangleq \frac{I_2}{I} = \frac{-z_{12}}{z_{22} + R} = \frac{y_{12}}{\Delta_Y R + y_{11}} = \frac{-1}{CR + D} \tag{5-111}$$

Finally, for the circuit of Fig. 5-19*d*,

$$Z_T \triangleq \frac{V_2}{I} = \frac{z_{12}R}{z_{11} + R} = \frac{-y_{22}R}{\Delta_Y R + y_{22}} = \frac{R}{A + RC} \tag{5-112}$$

The most important and most widely used circuit is the doubly terminated (or doubly loaded) reactance two-port, illustrated in Fig. 5-20. Depending on whether Thevenin or Norton model is used for the generator and whether V_2 or I_2 is used as output variable, any one of the four transfer functions A_V, A_I, Z_T, and Y_T can be used to describe the transmission properties of the circuit. If the circuit of Fig. 5-20*a* is chosen, for example, i.e., a Thevenin generator model, and V_2 as output variable, A_V is the proper transfer function. Now the branch relations are

$$V_1 = E - R_G I_1 \qquad V_2 = -I_2 R_L \tag{5-113}$$

They can be combined with the two-port relations (5-9) to give

$$(z_{11} + R_G)I_1 + z_{12}I_2 = E \qquad z_{12}I_1 + (z_{22} + R_L)I_2 = 0 \tag{5-114}$$

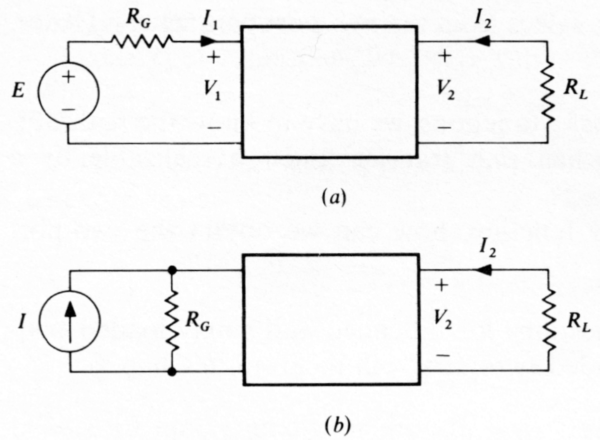


Figure 5-20 Doubly terminated two-ports.

Solving (5-114) for I_2 gives

$$I_2 = \frac{-z_{12}E}{\Delta_Z + z_{11}R_L + z_{22}R_G + R_G R_L} \tag{5-115}$$

Hence
$$A_V = \frac{V_2}{E} = \frac{-I_2 R_L}{E} = \frac{z_{12}R_L}{\Delta_Z + z_{11}R_L + z_{22}R_G + R_G R_L} \tag{5-116}$$

Carrying out the calculations in terms of the y_{ij}, that is, combining and solving (5-113) and (5-10), gives

$$A_V = \frac{-y_{12}R_L}{\Delta_Y R_G R_L + y_{11}R_G + y_{22}R_L + 1} \tag{5-117}$$

Finally, to express A_V in terms of the chain parameters, we combine and solve Eqs. (5-113) and (5-24). This results in

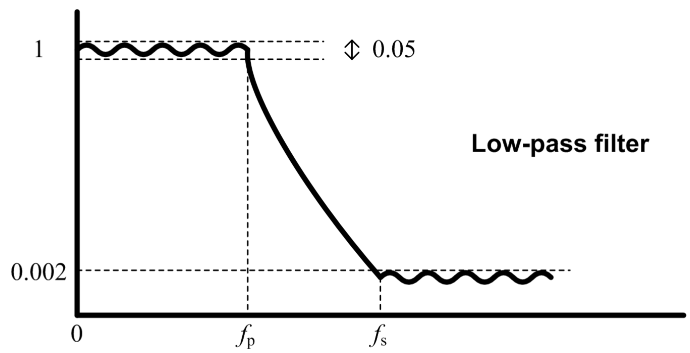
$$A_V = \frac{R_L}{AR_L + B + CR_G R_L + DR_G} \tag{5-118}$$

As will be shown in Chap. 6, the design of a doubly terminated two-port is expediently performed using a different transfer function $H(s)$, which is related to $A_V(s)$ by

$$H(s) \triangleq \frac{1}{2}\sqrt{\frac{R_L}{R_G}}\frac{E}{V_2} = \frac{\sqrt{R_L/R_G}}{2A_V} \tag{5-119}$$

Design of passive two-ports: filter, equalizer (gain or phase).

Delay line



Specs → Transfer functions → Z_{oc}, Y_{sc}, T, … → circuit

Example:

Calculate $Z_T = V_2 / I$ for the circuit shown using its z_{ij} parameters.

Solution:

Since $Z_T = z_{12}R / (z_{11} + R)$, we first find

$$z_{11} = (2s^2+1)/s(s^2+2) \approx 1/2s, s \to 0$$
$$z_{12} = (V_2/I_1) \text{ for } I_2 = 0 \to z_{12} = 1/(s^3 + 2s),\ s \to \infty$$

Substituting gives

$$Z_T = 1/(s^3 + 2s^2 + 2s + 1)$$

Checks: for s=0, $Z_T(s) = 1$; true from circuit diagram. For $s \to \infty$, $Z_T(s) \to s^{-3}$, also obvious from circuit.

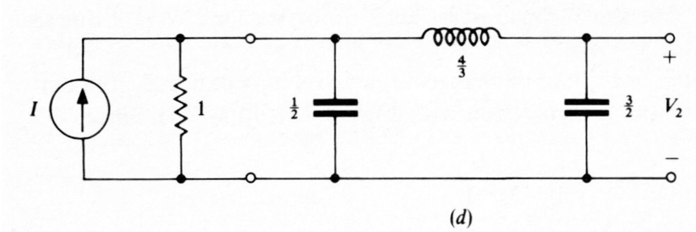


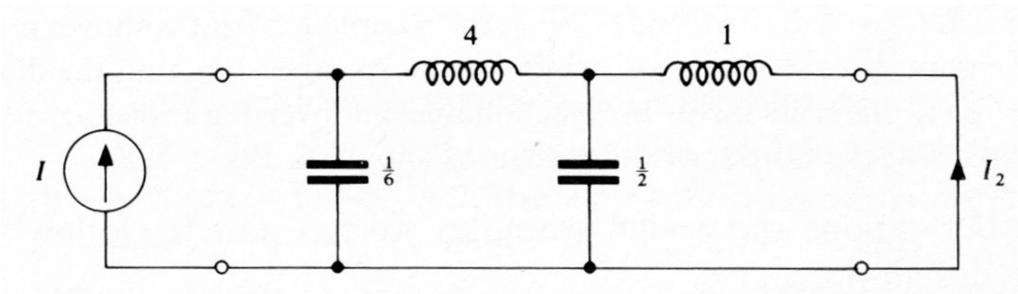
Figure 5-25 Reactance-ladder realization example.

Example:

Find the z_{ij} and AI for the circuit shown. The results are:

$$z_{12} = \frac{3}{s(s^2+2)} \qquad z_{22} = \frac{(s^2+1)(s^2+3)}{s(s^2+2)}$$

$$A_I \equiv \frac{I_2}{I} = -\frac{z_{12}}{z_{22}} = \frac{-3}{(s^2+1)(s^2+3)}$$



(c)

Again, testing for z=0 shows that $z_{22}= -z_{12} = 1/(2s/3)$ and AI=-1 are correct, as are $z_{22} \to s$, $z_{12} \to \frac{3}{s^3}$ and AI $\to \frac{-3}{s^4}$